Indian Statistical Institute

Mid-Semestral Examination 2016-2017

B.Math Third Year

Complex Analysis

Time : 3 Hours Date : 08.09.2016 Maximum Marks : 100 Instructor : Jaydeb Sarkar

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(i) Answer all questions. (ii) $B_r(z_0) := \{z \in \mathbb{C} : |z - z_0| < r\}$. (iii) $C_r(z_0) := \{z \in \mathbb{C} : |z - z_0| = r\}$. (iv) $U \subseteq \mathbb{C}$ open. (v) $Hol(U) = \{f : U \to \mathbb{C}$ holonorphic $\}$.

Q1. (15 marks) Evaluate:

$$(i) \int_{C_e(0)} \frac{ze^z}{z-i} dz, \quad (ii) \int_{C_e(0)} \frac{ze^z}{z-e^2 i} dz.$$

Q2. (10 marks) Let $\overline{B_1(0)} \subseteq U$ and $f(z) = \sum_{n=0}^{\infty} a_n z^n \in Hol(U)$ and that $|f(z)| \leq 1$ for all $z \in C_1(0)$. Prove that $|a_n| \leq 1$ for all n.

Q3. (10 marks) Let $\overline{B_r(0)} \subseteq U$ and $f \in Hol(U)$. Prove that

$$\int_{C_r(0)} \frac{f'(\zeta)}{\zeta - z} d\zeta = \int_{C_r(0)} \frac{f(\zeta)}{(\zeta - z)^2} d\zeta \qquad (\forall z \in B_r(0)).$$

Q4. (15 marks) Expand $\frac{1}{(1-z)^2}$ in a series of powers of z and find the radius of convergence.

Q5. (8+7 = 15 marks) Let $f \in Hol(\mathbb{C})$. What can you conclude about f:

(i) when $f(\mathbb{C}) \cap B_1(0)$ is an empty set.

(ii) when f, restricted to $\mathbb{R}$, is a 2π-periodic function.

Q6. (15 marks) Let $U \subseteq \mathbb{C}$ be a domain. Let $f, g \in Hol(U)$ and that $fg \equiv 0$. Prove that either $f \equiv 0$ or $g \equiv 0$.

Q7. (15 marks) Suppose f and g are continuous functions on $\overline{B_r(0)}$ and $f, g \in Hol(B_r(0))$ and that $f(z) \neq 0$ and $g(z) \neq 0$ for all $z \in \overline{B_r(0)}$. If $|f(z)| = |g(z)|$ for all $z \in C_r(0)$, then show that there exists a constant c such that $|c| = 1$ and $f(z) = cg(z)$ for all $z \in B_r(0)$.

Q8. (15 marks) Let $\Omega \subseteq \mathbb{R}^2$ be an open set and $u, v \in C^1(\Omega)$. Assume that u and v satisfy the Cauchy-Riemann equations in Ω. Assume moreover that $u(x,y)^2 + v(x,y)^2 \neq 0$ for all $(x,y) \in \Omega$. Show that the function

$$\frac{uu_x + vv_x}{u^2 + v^2}$$

is harmonic in Ω.

Q9. (10 marks) Prove that

$$f(z) := \int_0^1 \frac{\sin zt}{t} dt \qquad (z \in \mathbb{C}),$$

is an entire function.